

# DEFINABLE MAXIMAL DISCRETE SETS IN FORCING EXTENSIONS

DAVID SCHRITTESSER AND ASGER TÖRNQUIST

**ABSTRACT.** Let  $\mathcal{R}$  be a  $\Sigma_1^1$  binary relation, and recall that a set  $A$  is  $\mathcal{R}$ -discrete if no two elements of  $A$  are related by  $\mathcal{R}$ . We show that in the Sacks and Miller forcing extensions of  $L$  there is a  $\Delta_2^1$  maximal  $\mathcal{R}$ -discrete set. We use this to answer in the negative the main question posed in [5] by showing that in the Sacks and Miller extensions there is a  $\Pi_1^1$  maximal orthogonal family (“mof”) of Borel probability measures on Cantor space. A similar result is also obtained for  $\Pi_1^1$  mad families. By contrast, we show that if there is a Mathias real over  $L$  then there are no  $\Sigma_2^1$  mofs.

## 1. INTRODUCTION

(A) The present paper studies the definability of what in various contexts is called either *independent sets*, *orthogonal families*, or *antichains*. To capture these notions at once, we adopt the nomenclature of [18] and make the following definition:

**Definition 1.1.** Let  $\mathcal{R}$  be a binary relation on a set  $X$ . A set  $A \subseteq X$  is called  $\mathcal{R}$ -discrete iff

$$(\forall x, y \in A) \ x \neq y \implies \neg(x \mathcal{R} y).$$

By a *maximal  $\mathcal{R}$ -discrete set* we mean an  $\mathcal{R}$ -discrete set which is maximal under inclusion among  $\mathcal{R}$ -discrete sets.

The above definition is familiar in the context of graphs, i.e. symmetric irreflexive relations, where discrete sets are often also called *independent sets*.

Another situation in which maximal discrete sets are of interest is when  $\mathcal{R}$  is a *compatibility relation*, i.e. when  $\mathcal{R}$  is symmetric and reflexive. Such relations often arise from a preorder: if  $\preceq$  is a preorder, then the associated compatibility relation  $\mathcal{R}_{\preceq}$  is defined by

$$x \mathcal{R}_{\preceq} y \iff (\exists z) z \preceq x \wedge z \preceq y.$$

---

*Date:* Friday 30<sup>th</sup> October, 2015.

*2010 Mathematics Subject Classification.* 03E15, 03E35.

*Key words and phrases.* Descriptive set theory, consistency and independence results, discrete sets, analytic relations, maximal orthogonal families of measures, Sacks forcing, forcing, co-analytic sets, mad families.

The authors retain copyright.

In this context, an  $\mathcal{R}_{\preceq}$ -discrete set is often called an *antichain* for  $\preceq$ .

A straight-forward transfinite induction shows that maximal discrete sets always exist for any binary relation  $\mathcal{R}$ . However, the *definability* of such maximal discrete sets may be contentious. In Gödel's constructible universe  $L$  any  $\Sigma_1^1$  (i.e. *effectively* analytic) binary relation admits a  $\Delta_2^1$  maximal discrete set, a fact that follows routinely from the existence of a  $\Delta_2^1$  wellordering of the reals in  $L$  of order type  $\omega_1$ , with a good coding of initial segments. On the other hand, if we let  $\Gamma$  be the  $F_\sigma$  (in fact  $\Sigma_2^0$ ) graph on  $2^\omega$  where  $x \Gamma y$  iff  $x$  and  $y$  differ on exactly one bit, then a routine Baire category argument shows that  $\Gamma$  admits no Baire measurable maximal discrete set, and so by [10, Theorem 0.10] there is no  $\Delta_2^1$  maximal  $\Gamma$ -discrete set if there is a Cohen real over  $L$ . The situation is parallel with random reals.

The first goal of this paper is to show that the above failure does not persist in all forcing extensions of  $L$  with new reals.

**Theorem 1.2.** *Let  $\mathcal{R}$  be a  $\Sigma_1^1$  binary relation on an effectively presented Polish space, and let  $x$  be a Sacks or Miller real over  $L$ . Then there is a  $\Delta_2^1$  maximal  $\mathcal{R}$ -discrete set in  $L[x]$ .*

A suitably relativized version of the previous theorem applies more generally to  $\mathcal{R}$  which are  $\Sigma_1^1[a]$  for some real parameter  $a$ .

(B) Our main application of Theorem 1.2 is to the compatibility relation that comes from *absolute continuity* of Borel probability measures. Recall that if  $\mu$  and  $\nu$  are (non-trivial) measures on a measurable space  $X$ , then we write  $\mu \ll \nu$  just in case every set which is null for  $\nu$  is also null for  $\mu$ . Two measures  $\mu$  and  $\nu$  that are not compatible in  $\ll$  are called *orthogonal*, written  $\mu \perp \nu$ . By the Lebesgue decomposition theorem, for Borel probability measures this is equivalent to that there exists a Borel set  $A \subseteq X$  such that  $\nu(A) = 1$  and  $\mu(A) = 0$ .

Orthogonal families of measures in the Polish space  $P(X)$  of Borel probability measures on a Polish space  $X$  (see [14, Theorem 17.23, p.127]) show up in many different contexts, including representation theory, ergodic theory and operator algebras, see e.g. [21, 23]. Interest in the definability of maximal orthogonal families of measures (abbreviated *mofs*) can be traced back to the following question posed by Mauldin: If  $X$  is a perfect Polish space, is there an *analytic* maximal orthogonal family in  $P(X)$ ? The answer turns out to be no, as shown by Preiss and Rataj [20]. A new proof of this fact was provided by Kechris and Sofronidis [15] based on Hjorth's turbulence theory. Later, it was shown by Fischer and the second author [5] that if  $V = L$  then there is a  $\Pi_1^1$  (lightface) mof in  $P(2^\omega)$ . On the other hand, [5] and [3] established that if there is a Cohen or random real over  $L$  there are no  $\Pi_1^1$  mofs.

The seemingly restrictive nature of  $\Pi_1^1$  mofs motivated the following question in [5]: *If there is a  $\Pi_1^1$  mof in  $P(2^\omega)$ , must all reals be constructible?* Further compounding the intrigue, we will see in §2 below that no  $\Pi_1^1$  mof

contained in  $L$  remains maximal in any extension of  $L$  with new reals. Nevertheless, in this paper we will answer the above question in the negative by showing:

**Theorem 1.3.** *If  $x$  is a Sacks or Miller real over  $L$ , then in  $L[x]$  there is a  $\Pi_1^1$  mof in  $P(2^\omega)$ .*

A counterpoint to this is obtained in the last section, where the following is shown:

**Theorem 1.4.** *There are no  $\Pi_1^1$  mofs in the Mathias extension of  $L$ .*

Another application of Theorem 1.2 is to maximal almost disjoint families (“mad” families) of subsets of  $\omega$ . Such families are precisely the discrete sets for the compatibility relation that we get from the preorder  $\subseteq^*$ , inclusion modulo a finite set, in  $[\omega]^\omega$ . The study of the definability of mad families has a long history, see e.g. [17, 6, 4, 2, 25]. From Theorem 1.2 and [24] we get:

**Theorem 1.5.** *If  $x$  is a Sacks or Miller real over  $L$ , then in  $L[x]$  there is a  $\Pi_1^1$  mad family in  $[\omega]^\omega$ .*

(C) The paper is organized as follows: In §2 we show there can be no indestructible  $\Sigma_2^1$  mof in  $L[a]$ . In §3, after we review some well-known facts about Sacks and Miller forcing, we prove a slightly more general version of Theorem 1.2. We also list some general properties of forcings which allow the proof to go through. In §4, we apply this to mofs and show that if there is a  $\Sigma_2^1$  mof, there is a  $\Pi_1^1$  mof. §5 presents an argument that a Mathias real over  $L[a]$  rules out the existence of a  $\Sigma_2^1[a]$  mof. Using the same ideas, we sketch a new proof that there is no analytic mof. We close in §6 with open questions.

*Acknowledgements.* We would like to thank Stevo Todorčević for making us aware of the analogue of Galvin’s theorem for Miller forcing, which we use in the proof of Corollary 3.5.

The authors gratefully acknowledge the generous support from Sapere Aude grant no. 10-082689/FNU from Denmark’s Natural Sciences Research Council.

## 2. THERE IS NO INDESTRUCTIBLE $\Sigma_2^1$ MOF IN $L$

In this brief section we prove that there is no hope of finding a  $\Sigma_2^1$  mof in  $L$  that survives in an outer model which has new reals. The proof can be seen as be a warm-up for Theorem 5.1. The key property of  $\ll$  that we use is the so-called *ccc-below* property: If  $\mu \in P(X)$ , then any orthogonal family of measures  $\mathcal{F}$  such that  $\nu \ll \mu$  for all  $\nu \in \mathcal{F}$  must be countable.

For the following, we view  $P(2^\omega)$  as an effectively presented Polish space in precisely the manner described in [5].

**Theorem 2.1.** *Suppose  $L \models \text{“}\mathcal{A} \text{ is a } \Sigma_2^1 \text{ mof in } P(2^\omega)\text{”}$ , and suppose there is  $x \in \mathcal{P}(\omega) \setminus L$ . Then  $\mathcal{A}$  is not maximal (in  $V$ ).*

*Proof.* Note that for a  $\Sigma_2^1$  set  $\mathcal{A} \subseteq P(2^\omega)$ , the formula expressing ‘ $\mathcal{A}$  is orthogonal’ is  $\Pi_2^1$  and hence absolute. Thus, the theorem has the following equivalent form: if  $\mathcal{A}$  is a  $\Sigma_2^1$  mof in  $P(2^\omega)$  such that  $\mathcal{A} \subseteq L$ , then  $\mathcal{P}(\omega) \subseteq L$ .

For this, suppose  $\mathcal{A}$  is a  $\Sigma_2^1$  mof. From the product measure construction in [15, p. 1463] it follows easily that there is a  $\Pi_1^0$  Cantor subset of  $P(2^\omega)$  of pairwise orthogonal measures; let  $Y$  be such. Define

$$R = \{(\mu, \nu) \in Y \times \mathcal{A} : \mu \not\perp \nu\}.$$

Then  $R$  is  $\Sigma_2^1$  and we can find a  $\Sigma_2^1$  function  $F$  that uniformizes  $R$ . Since  $\mathcal{A}$  is maximal,  $F$  is a total function from  $Y$  to  $\mathcal{A}$ . Moreover, the ccc-below property of  $\ll$  implies that  $F$  is countable-to-one. The Mansfield-Solovay perfect set theorem (see e.g. [11, Theorem 25.23, p. 492]) now implies that all reals are constructible: Indeed for every  $\nu \in \mathcal{A}$ , the set  $F^{-1}(\nu)$  is  $\Sigma_2^1(\nu)$  and countable, and so contains only constructible reals. Thus all elements of  $Y$  are constructible, and since  $Y$  is  $\Delta_1^1$ -isomorphic to  $2^\omega$ , it follows that all reals are constructible.  $\square$

### 3. $\Delta_2^1$ MAXIMAL DISCRETE SETS IN THE SACKS OR MILLER EXTENSION

In this section, we prove Theorem 1.2 in the following, slightly stronger form:

**Theorem 3.1.** *Let  $x$  be a Sacks or Miller real over  $L[a]$ ,  $a \in \omega^\omega$ . For any  $\Sigma_1^1[a]$  binary relation  $\mathcal{R}$  on  $\omega^\omega$  there is a  $\Delta_2^1[a]$  predicate which defines a maximal  $\mathcal{R}$ -discrete set in both  $L[a]$  and  $L[a][x]$ .*

The theorem applies to arbitrary effectively presented Polish spaces, since any two uncountable such spaces are  $\Delta_1^1$  isomorphic. The argument also applies more generally to *arboreal* forcing notions satisfying certain conditions, which we list in Theorem 3.10.

Before delving into the proof, we collect a few preliminaries about Sacks forcing  $\mathbb{S}$  and Miller forcing  $\mathbb{M}$ . Firstly, we need the following elementary fact about descriptive complexity calculations and the forcing relation. Let  $\mathbb{P} \in \{\mathbb{S}, \mathbb{M}\}$ . In either case, we denote by  $\dot{x}_G$  the name for the generic real.

**Fact 3.2.** If  $\varphi(x, y)$  is a  $\Pi_1^1$  formula, the set

$$\{(p, a) \in \mathbb{P} \times \omega^\omega : p \Vdash_{\mathbb{P}} \varphi(\dot{x}_G, \check{a})\}$$

is  $\Pi_1^1$ .

*Proof.* We treat the case  $\mathbb{S}$  in detail. Clearly,  $p \Vdash_{\mathbb{S}} \varphi(\dot{x}_G, \check{a})$  if and only if the analytic set

$$A = \{x \in [p] : \neg \varphi(x, a)\} \tag{3.1}$$

is countable: if  $A$  is uncountable, then by the perfect set theorem there is a condition  $q \leq p$  with  $[q] \subseteq A$ , and  $q \Vdash_{\mathbb{S}} \varphi(\dot{x}_G, \check{a})$  by  $\Pi_1^1$ -absoluteness; if, on

the other hand,  $A$  is countable, then  $1_{\mathbb{S}} \Vdash \dot{x}_G \notin \check{A}$ , and by  $\Pi_1^1$ -absoluteness  $p \Vdash (\forall x \in [\check{p}] \setminus \check{A}) \varphi(x, \check{a})$ , proving the equivalence.

If  $A$  is countable, the effective perfect set theorem [19, Theorem 4F.1, p. 243] gives

$$\{x \in 2^\omega : \neg\varphi(x, a)\} \subseteq L_{\omega_1^a}[a].$$

In fact, the proof shows that there is a sequence  $(x_n)_{n \in \omega} \in L_{\omega_1^a}[a]$  such that

$$\{x \in 2^\omega : \neg\varphi(x, a)\} = \{x_n : n \in \omega\}.$$

Thus  $p \Vdash_{\mathbb{S}} \varphi(\dot{x}_G, \check{a})$  if and only if

$$(\exists (x_n)_{n \in \omega} \in \Delta_1^1[a])(\forall x) \neg\varphi(x, a) \implies (\exists n)x = x_n,$$

which is  $\Pi_1^1$  by [16, Corollary 4.19, p. 53].

For  $\mathbb{M}$ ,  $p \Vdash_{\mathbb{M}} \varphi(\dot{x}_G, \check{a})$  precisely if  $\{x \in 2^\omega : \neg\varphi(x, a)\}$  is contained in a  $K_\sigma$  set by [13] (or see [14, Corollary 21.23, p. 178]). The rest of the proof is analogous to the above.  $\square$

Secondly, we will use the following well-known fact to give us a practical way of talking about names for reals. For the sake of completeness, we include a proof. For the rest of this section, let  $A = \omega$  when  $\mathbb{P} = \mathbb{M}$ , and  $A = 2$  when  $\mathbb{P} = \mathbb{S}$ .

**Fact 3.3.** Sacks and Miller forcing have *continuous reading of names for reals*: If  $p \in \mathbb{P}$ ,  $\dot{x}$  is an  $\mathbb{P}$ -name and  $p \Vdash \dot{x} \in \omega^\omega$ , then there is  $\eta : A^\omega \rightarrow \omega^\omega$  continuous and  $q \leq p$  such that  $q \Vdash \eta(\dot{x}_G) = \dot{x}$ .

Any continuous function  $\eta : A^\omega \rightarrow \omega^\omega$  arises from a monotone map between trees  $\varphi : A^{<\omega} \rightarrow \omega^{<\omega}$ ; in the notation of [14, Definition 2.5, p. 7],  $\eta = \varphi^*$ . So we can regard the countable object  $\varphi$  as a ‘code’ for  $\eta$ . To say that  $\varphi$  gives rise to a total function is  $\Pi_1^1$ , whence absolute. We adopt the convention that, when considering the function  $\eta$  in a forcing extension, if  $\eta$  is coded by  $\varphi$  in this sense then  $\eta$  is always identified with the function defined by the code  $\varphi$ . Without this convention the statement itself of Fact 3.3 makes little sense.

*Proof of Fact 3.3.* For  $n \in \omega$ , the set  $D_n$  of  $p \in \mathbb{P}$  which decide a value for  $\dot{x}(\check{n})$  is dense and open. We construct  $q$  in a typical fusion argument: let  $p_0 = p$  and inductively find  $p_{n+1} \leq p_n$  such that  $p_{n+1} \cap A^n = p_n$  and whenever  $s \in A^n$ , then  $(p_{n+1})_s \in D_n$ , where  $(p_{n+1})_s = \{t \in p_{n+1} : s \subseteq t\}$ . Let  $q$  be the greatest lower bound of the sequence  $p_0, p_1, \dots$  (it exists as only finitely many changes occur on each  $A^n$ ).

For each  $n \in \omega$  there are sequences  $\{s_0^n, s_1^n, \dots\} \subseteq A^n$  and  $\{k_0^n, k_1^n, \dots\} \subseteq \omega$  such that  $\{s_0^n, s_1^n, \dots\} = q \cap A^n$  and

$$q \Vdash \check{s}_i^n \subseteq \dot{x}_G \implies \dot{x}(\check{n}) = \check{k}_i^n.$$

For  $r \in [q]$ , defining  $\eta(r)(n) = k_i^n$  whenever  $s_i^n \subseteq r$ , we have that  $\eta : [q] \rightarrow \omega^\omega$  is continuous and  $q \Vdash \eta(\dot{x}_G) = \dot{x}$ . We can easily extend  $\eta$  to a continuous function defined on all of  $A^\omega$ .  $\square$

Lastly, we need a Ramsey-theoretic statement, Corollary 3.5 below, which in the Sacks case follows from the following theorem due to Galvin (see [14, Theorem 19.7, p. 145]).

**Theorem 3.4** (Galvin's Ramsey theorem for Polish spaces). *Let  $X$  be a perfect Polish space, and suppose*

$$[X]^2 = P_0 \cup P_1$$

*is a partition of  $[X]^2$  into Baire measurable pieces. Then there is  $C \subseteq X$  perfect and  $i \in \{0, 1\}$  such that  $[C]^2 \subseteq P_i$ .*

Given a binary relation  $\mathcal{R}$ , a set  $A$  is called  $\mathcal{R}$ -complete iff

$$(\forall x, y \in A) \ x \neq y \implies x \mathcal{R} y.$$

**Corollary 3.5.** *Let  $\mathcal{R}$  be an analytic relation on a standard Borel space  $X$ , let  $\eta : A^\omega \rightarrow X$  be a Borel function and let  $p \in \mathbb{P}$ . Then there is  $q \in \mathbb{P}$  such that  $q \leq p$  and either  $\eta([q])$  is  $\mathcal{R}$ -complete, or  $\eta([q])$  is  $\mathcal{R}$ -discrete.*

*Proof.* For  $\mathbb{P} = \mathbb{S}$ , assume  $\mathcal{R}$  is symmetric and let  $X = [p]$  and

$$P_0 = \{\{x, y\} \in [X]^2 : \eta(x) \mathcal{R} \eta(y)\},$$

$P_1 = [X]^2 \setminus P_0$ . Let  $C$  be given by Galvin's theorem and pick  $q \in \mathbb{P}$  such that  $q \leq p$  and  $[q] = C$ . If  $[C]^2 \subseteq P_1$  then  $\eta(C)$  is  $\mathcal{R}$ -discrete and if  $[C]^2 \subseteq P_0$  then  $\eta(C)$  is  $\mathcal{R}$ -complete.

For  $\mathbb{P} = \mathbb{M}$ , as was pointed out by Stevo Todorćević, we may use [22, Corollary 5.68, p. 121] instead of Theorem 3.4 to derive the result.  $\square$

*Remark 3.6.* In the previous corollary, it is perfectly acceptable that  $\eta$  is constant, say. In that case  $\eta(2^\omega)$  is both  $\mathcal{R}$ -complete and  $\mathcal{R}$ -discrete at the same time.

**Definition 3.7.** For  $\mathcal{R}$  and  $\eta$  as in Corollary 3.5, call  $q \in \mathbb{P}$  a *Galvin witness* for  $\eta$  (and  $\mathcal{R}$ ) if  $\eta([q])$  is either  $\mathcal{R}$ -complete or  $\eta([q])$  is  $\mathcal{R}$ -discrete.

Note that  $\eta([q])$  being  $\mathcal{R}$ -discrete is  $\Pi_1^1[a]$  uniformly in  $\eta$  and  $q$ , and  $\eta([q])$  being  $\mathcal{R}$ -complete is  $\Pi_2^1[a]$ . In particular both are absolute for class models by Levy-Shoenfield, and thus so is the property of being a Galvin witness.

Now we are ready to prove the main theorem of this section:

*Proof of Theorem 3.1.* Let  $\mathcal{R} \subseteq (\omega^\omega)^2$  be  $\Sigma_1^1[a]$ ; the proof relativizes easily to the parameter  $a$ , so we suppress it below.

It suffices to produce a  $\Sigma_2^1$  formula  $\varphi$  which defines a maximal  $\mathcal{R}$ -discrete set in any  $\mathbb{P}$ -generic extension of  $L$ , since if  $\mathcal{A}$  is  $\Sigma_2^1$  and maximal  $\mathcal{R}$ -discrete set, then  $\mathcal{A}$  is in fact  $\Delta_2^1$ , since

$$x \notin \mathcal{A} \iff (\exists y \in \mathcal{A}) \ x \mathcal{R} y \wedge x \neq y.$$

Below we identify  $\mathbb{P} \times C(A^\omega, \omega^\omega)$  with a  $\Pi_1^1$  subset of  $\omega^\omega$ , by identifying both  $C(A^\omega, \omega^\omega)$  and  $\mathbb{P}$  with subsets of  $\omega^\omega$  (see the remark after Fact 3.3) and identifying  $\omega^\omega$  and  $(\omega^\omega)^2$  via some fixed effective bijection<sup>1</sup>.

Working in  $L$ , fix an enumeration  $\langle (p_\xi, \eta_\xi) : \xi < \omega_1 \rangle$  of  $\mathbb{P} \times C(A^\omega, \omega^\omega)$  such that

$$\xi < \delta \implies (p_\xi, \eta_\xi) <_L (p_\delta, \eta_\delta).$$

By recursion on  $\omega_1$ , we will define a sequence  $\langle q_\xi : \xi < \omega_1 \rangle$ , such that the following are satisfied:

- (i)  $q_\xi$  is a (not necessarily perfect) subtree of  $p_\xi$ .
- (ii) If  $p_\xi \not\Vdash (\forall \delta < \xi)(\forall y \in [q_\delta]) \eta_\xi(\dot{x}_G) \mathcal{R} \eta_\delta(y)$  then  $q_\xi = \emptyset$ .
- (iii) If (ii) fails and  $q \in \mathbb{S}$  is  $\leq_L$ -least such that  $q \leq p_\xi$  and

$$(\forall x \in [q])(\forall \delta < \xi)(\forall y \in [q_\delta]) \eta_\xi(x) \mathcal{R} \eta_\delta(y), \quad (3.2)$$

then

- (iv.a) if  $\eta_\xi([q])$  is  $\mathcal{R}$ -discrete then  $q_\xi = q$ ;
- (iv.b) if (iv.a) fails then  $q_\xi \subseteq q$  is the pruned subtree whose unique branch is the left-most branch of  $q$ .
- (iv) The set  $\mathcal{A}^0 = \{(q_\xi, \eta_\xi) : \xi < \omega_1\}$  is a  $\Sigma_2^1$ .

Above it is implicit in (iii) that  $q$  exists; this follows since in case (iii) the analytic set

$$\{x \in [p_\xi] : (\exists \delta < \xi)(\exists y \in [q_\delta]) \eta_\xi(x) \mathcal{R} \eta_\delta(y)\}$$

must be countable, since otherwise (ii) applies.

Suppose for now that  $\langle q_\xi : \xi < \omega_1 \rangle$  satisfies (i)–(iv) above. Then let  $\varphi(y)$  be the  $\Sigma_2^1$  formula

$$(\exists q, \eta)(\exists x \in [q])(q, \eta) \in \mathcal{A}^0 \wedge y = \eta(x).$$

Clause (iii) ensures that  $\varphi$  defines an  $\mathcal{R}$ -discrete set in any model. For maximality, suppose, seeking a contradiction, that

$$p \Vdash (\exists x \in \omega^\omega)(\neg \varphi(x) \wedge (\forall y)(\varphi(y) \implies x \mathcal{R} y)).$$

Then there is a total continuous function  $\eta : A^\omega \rightarrow \omega^\omega$  such that

$$p \Vdash \neg \varphi(\eta(\dot{x}_G)) \wedge (\forall y)(\varphi(y) \implies \eta(\dot{x}_G) \mathcal{R} y). \quad (*)$$

By Corollary 3.5, we may assume that  $p$  is a Galvin witness for  $\eta$ , since we otherwise can replace  $p$  by a stronger condition.

Let  $\delta$  be such that  $(p, \eta) = (p_\delta, \eta_\delta)$ . Then clause (ii) fails for  $p_\delta$ , and so clause (iii) applies. Let  $q \leq p_\delta$  be as in clause (iii).

If  $\eta_\delta[q]$  is  $\mathcal{R}$ -discrete then  $q_\delta = q$ . Since  $q \Vdash x_G \in [q]$  it follows that  $q \Vdash \eta_\delta(\dot{x}_G) \in \mathcal{A}$ , contradicting that  $p \Vdash \neg \varphi(\eta(\dot{x}_G))$ .

---

<sup>1</sup>For the case  $A = 2$ , we could alternatively use that the set of continuous functions  $C(2^\omega, \omega^\omega)$  has an effective presentation as a Polish metric space, and so we can regard it as a  $\Pi_2^0$  subset of  $2^\omega$ .

So it must be that  $\eta([q])$  is not  $\mathcal{R}$ -discrete, and so clause (iv.b) applies. Since  $p$  is a Galvin witness for  $\eta$ , it follows that  $\eta([p])$  is  $\mathcal{R}$ -complete. Let  $z \in [q_\delta]$  be the unique branch through  $q_\delta$ . Then

$$(\forall x \in [p]) \eta(x) = \eta(z) \vee \eta(x) \mathcal{R} \eta(z),$$

and since this is  $\Pi_2^1[z]$ , it follows by Shoenfield's absoluteness theorem that  $p \Vdash \eta(\dot{x}_G) = \eta(z) \vee \eta(\dot{x}_G) \mathcal{R} \eta(z)$ , contradicting  $(*)$ .

It is routine that  $\langle q_\xi : \xi < \omega_1 \rangle$  satisfying (i)–(iii) above can be found. In fact, (i)–(iii) determine the sequence  $\langle q_\xi : \xi < \omega_1 \rangle$  uniquely. So proving the following claim will finish the proof.

**Claim 3.8.** *The set  $\mathcal{A}^0 = \{(q_\xi, \eta_\xi) : \xi < \omega_1\}$  is  $\Sigma_2^1$ .*

*Proof of claim:* The set of sequences  $\vec{x} = \langle (p_n^*, \eta_n^*) : n \in \alpha \rangle$ , for  $\alpha \leq \omega$ , such that

$$(\exists \xi < \omega_1) \{(p_n^*, \eta_n^*) : n \in \alpha\} = \{(p_\delta, \eta_\delta) : \delta < \xi\}. \quad (3.3)$$

is  $\Sigma_2^1$ ; since  $\leq_L$  is a strongly  $\Delta_2^1$  well-ordering of  $\omega^\omega$  (see [8] or [26]) and  $\mathbb{P} \times C(A^\omega, \omega^\omega)$  was identified with a  $\Pi_1^1$  subset of  $\omega^\omega$ , this follows easily by the proof of [19, Exercise 5A.1, p. 287]. Let  $\Psi(\vec{x})$  be a  $\Sigma_2^1$  formula equivalent to (3.3).

Now assume  $\Psi(\vec{x})$  holds,  $\vec{y} = \langle q_n^* : n \in \alpha \rangle$  enumerates  $\{q_\delta : \delta < \xi\}$ , and suppose for all  $\delta < \xi$  we have  $p_n^* = p_\delta \iff q_n^* = q_\delta$ .

Observe that (ii) and (iii) are  $\Sigma_2^1$  uniformly in  $(\vec{x}, \vec{y}, p_\xi, \eta_\xi, q_\xi)$ : (ii) is  $\Pi_1^1$  in the parameters  $(\vec{x}, \vec{y}, p_\xi, \eta_\xi)$  by 3.2. The property expressed in (3.2) is  $\Pi_1^1$  in  $(\vec{x}, \vec{y}, \eta_\xi, q)$ . Thus, that  $q$  be minimal with that property is  $\Sigma_2^1$  in these parameters, as  $\leq_L$  is strongly  $\Delta_2^1$  (by ‘closure under bounded quantification’). Clauses (iv.a) and (iv.b) are Boolean combinations of  $\Sigma_1^1$  formulas in the parameters  $(\eta_\xi, q_\xi, q)$  by the remarks following Definition 3.7. Thus, (iii) is easily seen to be  $\Sigma_2^1$  in  $(\vec{x}, \vec{y}, p_\xi, \eta_\xi, q_\xi)$ .

So we may express the conjunction of (ii) and (iii) by a  $\Sigma_2^1$  formula

$$\Theta(q, p_\xi, \eta_\xi, \vec{x}, \vec{y}),$$

i.e. for  $\vec{x}$  and  $\vec{y}$  as above,  $\Theta(q, p_\xi, \eta_\xi, \vec{x}, \vec{y})$  holds if and only if  $q = q_\xi$ .

Thus  $\vec{y} = \langle q_n^* : n \in \alpha \rangle$ , with  $\alpha \leq \omega$ , enumerates an initial segment of  $\langle q_\xi : \xi < \omega_1 \rangle$  exactly if the following formula holds:

$$(\exists \vec{x}) \Psi(\vec{x}) \wedge [(\forall n \in \alpha) \Theta(q_n^*, p_n^*, \eta_n^*, \vec{x} \upharpoonright n, \vec{y} \upharpoonright n)]$$

where  $\vec{x} \upharpoonright n = \langle (p_m^*, \eta_m^*) : m \in \alpha, (p_m^*, \eta_m^*) \leq_L (p_n^*, \eta_n^*) \rangle$  (similarly for  $\vec{y} \upharpoonright n$ ). The formula above is easily seen to be equivalent to a  $\Sigma_2^1$  formula; it follows that  $\mathcal{A}^0$  is  $\Sigma_2^1$ . Claim.  $\dashv$

□

We also get the following effective corollary for  $\Sigma_1^1$  relations:

**Corollary 3.9.** *Under the hypothesis of Theorem 3.1, if  $a' \in L[x]$  and  $\mathcal{R}$  is a  $\Sigma_1^1[a']$  binary relation on a effectively presented Polish space  $X$ , then in  $L[x]$  there is a  $\Delta_2^1[a']$  maximal  $\mathcal{R}$ -discrete set in  $X$ .*



*Proof.* We may assume  $X = \omega^\omega$ . If  $a' \in L$ , then Theorem 3.1 gives a  $\Delta_2^1[a']$  formula  $\varphi$  defining a maximal  $\mathcal{R}$ -discrete set in  $L[x]$ . If  $a' \notin L$  then  $L[x] = L[a']$  (as  $x$  is of minimal degree, see [7]), and Theorem 3.1 provides a  $\Delta_2^1[a']$  formula  $\varphi$  defining a maximal  $\mathcal{R}$ -discrete set in  $L[a']$ —and, incidentally, its Sacks (resp. Miller) extension—starting from the strongly  $\Delta_2^1[a']$  well-ordering of  $L[a']$ .  $\square$

It's simple to axiomatize a class of forcings for which the above proof goes through. A forcing  $\mathbb{P}$  is *arboreal* if and only if its conditions are perfect trees on  $A$  where  $A \subseteq \omega$ , ordered by inclusion. Any extension of  $V$  by a  $(\mathbb{P}, V)$ -generic filter  $G$  is generated by the single ‘generic’ real  $\bigcap_{p \in G} [p]$ ; its name we denote by  $\dot{x}_G^\mathbb{P}$ . A real is called  $(\mathbb{P}, V)$ -generic over if and only if it arises in this way. For example, Sacks, Miller, Mathias and Laver are (equivalent to) arboreal forcings (see e.g. [1] and [7]).

**Theorem 3.10.** *Let  $\mathbb{P}$  be an arboreal forcing such that:*

- (A)  $\mathbb{P}$  has Borel reading of names (in the sense of [27, Proposition 2.3.1, p. 29]).
- (B) If  $\varphi(x, y)$  is a  $\Pi_1^1$  predicate then  $\{(p, a) \in \mathbb{P} \times \omega^\omega : p \Vdash \varphi(\dot{x}_G^\mathbb{P}, \check{a})\}$  is  $\Delta_2^1$ .
- (C) The analogue of Galvin’s theorem holds for  $\mathbb{P}$ : for  $\mathcal{R}$  as in Theorem 3.1, a Borel function  $\eta: A^\omega \rightarrow \omega^\omega$  and  $p \in \mathbb{P}$ , there is  $q \in \mathbb{P}$ ,  $q \leq p$  such that  $q$  is a Galvin witness for  $\mathcal{R}$  and  $\eta$ .

*Then the analogue of Theorem 3.1 holds when  $x$  is a  $(\mathbb{P}, L[a])$ -generic real.*

We mention without proof that (B) can be replaced by: for all countable transitive  $M$  and  $p \in \mathbb{P} \cap M$  there is  $q \leq p$  s.t. any  $r \in [q]$  is a  $(\mathbb{P}, M)$ -generic.

*Proof of Theorem 3.10.* One difference to the proof of Theorem 3.1 is how we obtain the enumeration  $\langle p_\xi, \eta_\xi : \xi < \omega_1 \rangle$  at the beginning. The second coordinate now has to enumerate all codes for total Borel functions; the set of such codes is  $\Pi_1^1$  (to see this, observe that if  $f$  is  $\Delta_1^1[a]$  then  $f(x)$  is  $\Delta_1^1[x, a]$ ; now use [16, Corollary 4.19, p. 53]), so the proof goes through as before. It remains to notice that when even just requiring (B), clauses (ii) and (iii) remain  $\Sigma_2^1$ .  $\square$

#### 4. A CO-ANALYTIC MOF IN THE MILLER AND SACKS EXTENSIONS

**Theorem 4.1.** *If  $x$  is a Miller or Sacks real over  $L[a]$ , then*

$$L[a][x] \models \text{“there is a } \Pi_1^1[a] \text{ mof in } P(2^\omega)\text{”}.$$

This follows immediately from Theorem 1.2 together with the following lemma.

**Lemma 4.2.** *If there is a  $\Sigma_2^1[a]$  mof in  $P(2^\omega)$ , then there is a  $\Pi_1^1[a]$  mof.*

*Proof.* We suppress the parameter  $a$  below.

The proof is based on a slight simplification of the coding method from [5]. Let  $P_c(2^\omega)$  denote the set of *atomless* Borel probability measures on  $2^\omega$ , i.e.  $\mu \in P(2^\omega)$  such that  $\mu(\{x\}) = 0$  for any  $x$ . This set is  $\Pi_2^0$  as a subset of  $P(2^\omega)$ , see [5, Lemma 2.1]. Given  $\mu \in P_c(2^\omega)$ , let  $y$  be the left-most branch of the tree

$$\{t \in 2^{<\omega} : \mu(N_{y \upharpoonright n})\},$$

where  $N_s = \{x \in 2^\omega : s \subseteq x\}$  is the basic open neighborhood determined by  $s \in \omega^n$ .

For  $\mu \in P_c(2^\omega)$  given, let  $n(0), n(1), \dots$  enumerate the infinite set of  $n$  such that  $\mu(N_{y \upharpoonright n \smallfrown 0}) > 0$  and  $\mu(N_{y \upharpoonright n \smallfrown 1}) > 0$  and define  $G(\mu) \in 2^\omega$  by

$$G(\mu)(i) = \begin{cases} 0 & \text{if } \mu(N_{y \upharpoonright n(i) \smallfrown 0}) \geq \mu(N_{y \upharpoonright n(i) \smallfrown 1}) \\ 1 & \text{otherwise.} \end{cases}$$

We say  $G(\mu)$  is “coded” by  $\mu$ . As in [5], we can find a  $\Delta_1^1$  coding function  $F: P_c(2^\omega) \times 2^\omega \rightarrow P_c(2^\omega)$  such that for all  $\mu \in P_c(2^\omega)$  and  $y \in 2^\omega$ ,  $F(\mu, y)$  is absolutely equivalent to  $\mu$  and codes  $y$ , that is,  $G(F(\mu, y)) = y$ .

Now let  $\mathcal{A}$  be a  $\Sigma_2^1$  mof. By possibly modifying  $\mathcal{A}$  slightly, we may assume that  $\mathcal{A} \cap P_c(2^\omega)$  is a maximal orthogonal family among the atomless measures. Let  $R$  be  $\Pi_1^1$  such that  $\mu \in \mathcal{A} \iff (\exists y) R(\mu, y)$ . By  $\Pi_1^1$  uniformization, we can assume  $R$  is a functional relation, i.e.

$$(\forall x \in \text{dom}(R)) (\exists! y) R(x, y).$$

Fix a  $\Sigma_1^0$  bijection  $(x, y) \mapsto x \oplus y$  from  $(2^\omega)^2$  to  $2^\omega$ , and let  $x \mapsto (x)_i$ , for  $i \in \{0, 1\}$  be the pair of maps such that for all  $z \in 2^\omega$ ,  $z = (z)_0 \oplus (z)_1$  (i.e. the components of the inverse of our bijection). Let  $g: 2^\omega \rightarrow P_c(2^\omega)$  be a  $\Delta_1^1$  bijection.

Define  $\mathcal{A}' \subseteq P_c(2^\omega)$  by letting  $\mu' \in \mathcal{A}'$  just in case  $\mu' \in P_c(2^\omega)$  and

$$(\forall z, \mu, y) [z = G(\mu') \wedge \mu = g((z)_0) \wedge y = (z)_1] \implies \mu' = F(\mu, z) \wedge R(\mu, y). \quad (4.1)$$

Then  $\mathcal{A}'$  is a maximal orthogonal family of measures in  $P_c(2^\omega)$ , since every  $\mu' \in \mathcal{A}'$  is of the form  $F(\mu, z)$  for some  $\mu \in \mathcal{A}$  and  $z \in 2^\omega$ .

Clearly  $\mathcal{A}'$  is  $\Pi_1^1$ . By enlarging  $\mathcal{A}'$  to contain all Dirac measures (i.e. measures concentrating on a single point), we obtain a  $\Pi_1^1$  mof.  $\square$

## 5. NO $\Pi_1^1$ MOFS IN THE MATHIAS EXTENSION

The purpose of this section is to complement Theorem 4.1 by showing that its conclusion fails when  $x$  is a Mathias real over  $L$ .

**Theorem 5.1.** *If  $x$  is a Mathias real over  $L[a]$ , then*

$$L[a][x] \models \text{“there is no } \Sigma_2^1[a] \text{ mof”}.$$

In the process of proving this we will also obtain a new proof that there are no analytic mofs, see the end of this section.

The proof of Theorem 5.1 requires multiple steps. We start by defining a way of assigning to each  $x \in [\omega]^\omega$  a product measure on  $2^\omega$ . Let  $f_x : \omega \rightarrow \omega$  denote the unique increasing function such that  $x = f_x[\omega]$ . Then define a sequence  $\alpha^x \in [\frac{1}{4}, \frac{3}{4}]^\omega$  by

$$\alpha_n^x = \begin{cases} \frac{1}{4} + \frac{1}{2\sqrt{f_x^{-1}(n)+1}} & n \in x, \\ \frac{1}{4} & n \notin x, \end{cases}$$

and define  $\mu^x \in P(2^\omega)$  by

$$\mu^x = \prod_{n \in \omega} (\alpha_n^x \delta_0 + (1 - \alpha_n^x) \delta_1),$$

where  $\delta_i \in P(\{0, 1\})$  is the Dirac measure concentrated at  $i \in \{0, 1\}$ . For  $x, y \in [\omega]^\omega$ , let

$$\rho(x, y) = \sum_{n \in \omega} (\alpha_n^x - \alpha_n^y)^2.$$

Note that if  $z \subseteq y \subseteq x$  then  $\rho(y, x) \leq \rho(z, x)$ .

The intention behind the definition of  $\alpha^x$  is to be able to use Kakutani's theorem on equivalence and orthogonality of product measures. Specifically, [12, Corollary 1, p. 222] gives

$$\mu^x \sim \mu^y \iff \rho(x, y) < \infty \tag{5.1}$$

and

$$\mu^x \perp \mu^y \iff \rho(x, y) = \infty.$$

Let then

$$\mathcal{F} = \{g \in [\omega]^\omega : \rho(g, \omega) < \infty\},$$

and define a binary operation  $\cdot$  on  $[\omega]^\omega$  by

$$x \cdot y = f_y \circ f_x[\omega].$$

We can think of the operation  $x \cdot y$  as follows:  $f_y$  identifies  $\omega$  and  $y$ , and  $x \cdot y$  is the copy of  $x$  inside of  $y$  under this identification.

**Proposition 5.2.**

(i) For all  $x, y, z \in [\omega]^\omega$

$$\sqrt{\rho(x, y)} \leq \sqrt{\rho(x, z)} + \sqrt{\rho(z, y)}$$

holds, even if  $\rho(x, y)$  is infinite. In particular,  $\sqrt{\rho}$  is finite on  $\mathcal{F}$  and defines a complete metric on  $\mathcal{F}$ , inducing a Polish topology on  $\mathcal{F}$ . In this topology,  $\mathcal{F}$  is a perfect Polish space.

(ii) The operation  $\cdot$  is associative and makes  $[\omega]^\omega$  a monoid with the unit being  $\omega$ .

(iii) For all  $g \in \mathcal{F}$  and  $x \in [\omega]^\omega$  we have  $\rho(g \cdot x, x) = \rho(g, \omega)$  and  $\mu^{g \cdot x} \sim \mu^x$ . It follows that  $\mathcal{F}$  is closed under the operation  $\cdot$ , and so is a monoid with unit  $\omega$ .

(iv) If  $|x \triangle y| < \infty$  then  $\rho(x, y) < \infty$ .

(v) For any  $g \in \mathcal{F}$  and  $k \in \omega$ , it holds that  $g \cdot (\omega \setminus \{k\}) \in \mathcal{F}$ , and

$$\lim_{k \rightarrow \infty} \rho(g \cdot (\omega \setminus \{k\}), g) = 0.$$

(vi) For any  $x \in [\omega]^\omega$ , the equivalence relation  $\sim^x$ , defined in  $[x]^\omega$  by  $z \sim^x y \iff \mu^z \sim \mu^y$ , has meagre, dense classes in the Polish topology on  $[x]^\omega$ .

*Proof.* (i) Note that  $\sqrt{\rho(x, y)} = \|\alpha^x - \alpha^y\|_2$ , where  $\|\cdot\|_2$  is the 2-norm. The norm inequality

$$\|\alpha^x - \alpha^y\|_2 \leq \|\alpha^x - \alpha^z\|_2 + \|\alpha^z - \alpha^y\|_2$$

holds in the strong sense that if the left hand side is infinite, then so is the right hand side (use that  $\ell_2(\omega)$  is closed under addition). This establishes the inequality in (i). The map  $g \mapsto \alpha^g - \alpha^\omega$  is then an isometric embedding of  $\mathcal{F}$  into the Hilbert space  $\ell_2(\omega)$ . It is straight-forward to check that the image under  $g \mapsto \alpha^g - \alpha^\omega$  is closed in  $\ell_2(\omega)$ . Finally, (v) implies that  $\mathcal{F}$  has no isolated points.

(ii) follows immediately from the definitions. (iii) is follows easily from (i), eq. (5.1), and the definition of  $\rho$ . For (iv) and (v) we need:

**Claim:**  $\omega \setminus k \in \mathcal{F}$  for any  $k \in \omega$ .

*Proof of Claim:*

$$\sum_{n \geq k} (\alpha_n^\omega - \alpha_n^{\omega \setminus k})^2 = \sum_{n \geq k} \left( \frac{1}{\sqrt{n+1}} - \frac{1}{\sqrt{n+1-k}} \right)^2,$$

and the right hand side above converges since, using a small amount of calculus, we have  $(n-k)^{-\frac{1}{2}} - n^{-\frac{1}{2}} \leq n^{-\frac{3}{2}}$  for  $n > k$  sufficiently large. Claim.  $\dashv$

(iv) Suppose  $|x \triangle y| < \infty$  and let  $z = x \cup y$ . Let  $k$  be such that  $z \setminus k \subseteq x \cap y$  and  $k_0 = |z \cap k|$ . Then

$$\rho(z, x), \rho(z, y) \leq \rho(z, x \cap y) \leq \rho(z, z \setminus k) = \rho(z, (\omega \setminus k_0) \cdot z) < \infty,$$

with the last inequality following from the previous claim and (iii).

(v) Since  $\omega \setminus \{k\} \supseteq \omega \setminus (k+1)$  the claim gives  $\omega \setminus \{k\} \in \mathcal{F}$  for all  $k \in \omega$ . Now

$$\rho(g \cdot (\omega \setminus \{k\}), g) = \sum_{n \geq k} (\alpha_n^{g \cdot (\omega \setminus \{k\})} - \alpha_n^g)^2 = \sum_{n \geq k} (\alpha_n^{g \cdot (\omega \setminus \{0\})} - \alpha_n^g)^2,$$

and the last sum tends to 0 as  $k \rightarrow \infty$  since  $\rho(g \cdot (\omega \setminus \{0\}), g) < \infty$ .

(vi) By eq. (5.1) the relation  $\sim^x$  is  $F_\sigma$ . (iv) immediately gives that all  $\sim^x$  classes are dense in  $[x]^\omega$ . On the other hand, it is easy to check that if  $y \in [x]^\omega$  and the complement of  $f_x^{-1}(y)$  is not in the summable ideal, then  $\rho(y, x) = \infty$ . So  $\sim^x$  has at least two (necessarily dense) classes. It follows that the complement of any  $\sim^x$  class is a dense  $G_\delta$  set in  $[x]^\omega$ , whence each  $\sim^x$  class is meagre in  $[x]^\omega$ .  $\square$

*Remark 5.3.* (v) in the previous proposition intends to say that multiplication on the right in the monoid  $\mathcal{F}$  has at least *some* amount of continuity at  $\omega$  (the identity). By contrast, (iii) shows that left multiplication in  $\mathcal{F}$  is continuous at  $\omega$ . We do not know if right multiplication is actually continuous at the identity, but it seems unlikely.

Let  $X$  be a Polish space. Recall that the equivalence relation  $F_2^X$  on  $X^\omega$  is defined by

$$\vec{x} F_2^X \vec{y} \iff \{\vec{x}_n : n \in \omega\} = \{\vec{y}_n : n \in \omega\}.$$

**Lemma 5.4.** *Let  $\vartheta : [\omega]^\omega \rightarrow X^\omega$  be a continuous function (w.r.t. the Polish topologies), and suppose  $\vartheta$  is  $(\mathcal{F}, F_2^X)$ -equivariant, that is,*

$$(\forall g \in \mathcal{F})(\forall x \in [\omega]^\omega) \vartheta(g \cdot x) F_2^X \vartheta(x).$$

*Then there is a non-empty open set  $U_0$  such that  $\vartheta_0 \upharpoonright U_0$  is constant, where, in general,  $\vartheta_l$  is defined by  $\vartheta_l(x) = \vartheta(x)_l$  for  $l \in \omega$ .*

*Proof* (à la Hjorth). For the purpose of this proof, we identify  $[\omega]^\omega$  with a  $G_\delta$  subset of  $2^\omega$  in the natural way. Define for each  $l \in \omega$  a closed set

$$A_l = \{(g, x) \in \mathcal{F} \times [\omega]^\omega : \vartheta_l(g \cdot x) = \vartheta_0(x)\},$$

and let  $l_0$  be least such that  $A_{l_0}$  is non-meagre; this exists because  $\mathcal{F} \times [\omega]^\omega = \bigcup_{l \in \omega} A_l$ . Let  $V \times U \subseteq A_{l_0}$  be open and non-empty, and fix  $g_0 \in V$ . Using (v) of Proposition 5.2, find  $k_0$  such that

$$(\forall k > k_0) g_0 \cdot (\omega \setminus \{k\}) \in V.$$

Let  $s_0 \in 2^{<\omega}$  be such that  $N_{s_0} \subseteq U$ . By either making  $s_0$  longer or  $k_0$  larger, we may assume that  $s_0$  is the characteristic function of a set with  $k_0$  elements.

**Claim:** If  $y, z \in N_{s_0}$  differ on only one bit then  $\vartheta_0(y) = \vartheta_0(z)$ .

*Proof of Claim:* Suppose  $n \in y$  and  $n \notin z$ . Let  $k = f_y^{-1}(n)$ , and note that  $z = (\omega \setminus \{k\}) \cdot y$  and  $k > k_0$ . Since  $g_0 \cdot (\omega \setminus \{k\}) \in V$  we have

$$\vartheta_0(z) = \vartheta_0(\omega \setminus \{k\} \cdot y) = \vartheta_{l_0}(g_0 \cdot (\omega \setminus \{k\}) \cdot y) = \vartheta_0(y).$$

Claim.  $\dashv$

Now  $U_0 = N_{s_0}$  works, since the claim implies that the continuous function  $\vartheta_0$  is constant on a dense set in  $N_{s_0}$ .  $\square$

*Proof of Theorem 5.1.* Work in  $L[x]$ , where  $x$  is a Mathias real of  $L$ . By Lemma 4.2, it is enough to show that there is no  $\Pi_1^1$  mof. (For notational convenience, we suppress the parameter  $a$ .)

Suppose for a contradiction that  $\mathcal{A} \subseteq P(2^\omega)$  is a  $\Pi_1^1$  mof. Let  $Q \subseteq [\omega]^\omega \times P(2^\omega)^\omega$  be

$$Q = \{(x, (\nu_n)) : (\forall n)(\nu_n \in \mathcal{A} \wedge \nu_n \not\perp \mu^x) \wedge (\forall \mu)(\mu \not\perp \mu^x \longrightarrow (\exists n)\nu_n \not\perp \mu)\}.$$

(Thus  $(x, (\nu_n)) \in Q$  iff  $(\nu_n)$  enumerates the countably many measures in  $\mathcal{A}$  that are not orthogonal to  $\mu^x$ .) Since  $\mathcal{A}$  is maximal the sections  $Q_x$  are never empty, and by the  $\Pi_1^1$  uniformization theorem, we can find a function  $\vartheta : [\omega]^\omega \rightarrow P(2^\omega)^\omega$  which has a  $\Pi_1^1$  graph, and such that  $(x, \vartheta(x)) \in Q$  for all  $x \in [\omega]^\omega$ . Note that if  $\mu^x \simeq \mu^y$  then

$$\{\vartheta_n(x) : n \in \omega\} = \{\vartheta_n(y) : n \in \omega\},$$

so that by (iii) of Proposition 5.2,  $\vartheta$  is  $(\mathcal{F}, F_2^{P(2^\omega)})$ -equivariant.

It is easy to check that then  $\vartheta^{-1}(U)$  is  $\Delta_2^1$  for every basic open set  $U \subseteq P(2^\omega)^\omega$ . Since  $x$  is a Mathias real over  $L$ , every  $\Delta_2^1$  set is completely Ramsey (by [10, Theorem 0.9]). A standard argument [14, Exercise 19.19, p. 134] then shows that there is  $x \in [\omega]^\omega$  such that  $\vartheta \upharpoonright [x]^\omega$  is continuous (w.r.t. the Polish topology on  $[x]^\omega$ ).

From Lemma 5.4 it follows that there is a non-empty open set  $U \subseteq [x]^\omega$  and  $\nu \in P(2^\omega)$  such that  $\vartheta_0(x) = \nu$  for all  $x \in U$ . By (vi) of Proposition 5.2 there is an uncountable (indeed a perfect) set  $P \subseteq U$  such that if  $x, y \in P$  and  $x \neq y$ , then  $\mu^x \perp \mu^y$ . Now for every  $x \in P$  we have  $\mu^x \not\perp \nu$ , contradicting the ccc-below property of  $\ll$ .  $\square$

The above line of argument also gives a new proof of the theorem of Preiss and Rataj, which we sketch below. Unlike the new proof that was given by Kechris and Sofronidis in [15], the proof below does not rely directly on Hjorth's turbulence theory. All the same, Lemma 5.4 above owes a debt to [9, Lemma 3.14, p. 42] that can scarcely be ignored.

**Theorem 5.5** ([20]). *There are no analytic mofs in  $P(2^\omega)$ .*

*Sketch of proof.* Suppose  $\mathcal{A}$  were an analytic mof. By maximality,  $\mathcal{A}$  would be Borel, and maximality along with the ccc-below property gives that the Borel set

$$Q' = \{(x, \nu) \in [\omega]^\omega \times P(2^\omega) : \mu^x \not\perp \nu \wedge \nu \in \mathcal{A}\}$$

would have all vertical sections  $Q'_x$  non-empty and countable. Then we could find countably many Borel functions  $\vartheta_l : [\omega]^\omega \rightarrow P(2^\omega)$  such that

$$Q' = \bigcup_{l \in \omega} \text{graph}(\vartheta_l).$$

The equivariance

$$\mu^x \sim \mu^y \implies \{\vartheta_n(x) : n \in \omega\} = \{\vartheta_n(y) : n \in \omega\}$$

is clear. Again, [14, Exercise 19.19, p.134] would allow us to find  $x \in [\omega]^\omega$  such that  $\vartheta_l \upharpoonright [x]^\omega$  is continuous (w.r.t. the Polish topology) for all  $l$ . A contradiction is then obtained in exactly the same way it was in the proof of Theorem 5.1.  $\square$

## 6. OPEN PROBLEMS

Given the results of this paper, we pose the following questions:

- (1) Does the analogue of Theorem 3.1 hold for the Laver extension?  
(Note that the analogue of Galvin's theorem is false for Laver forcing.)
- (2) Is there a model with a  $\Pi_1^1$  mof such that in addition, for any  $r \in 2^\omega$ , there is a Sacks (alternatively, a Miller) real over  $L[r]$ ?
- (3) Is the existence of a  $\Pi_1^1$  mof consistent with  $2^\omega = \omega_2$  or even  $2^\omega = \omega_3$ ?
- (4) Does Theorem 3.1 fail for  $\mathcal{R}$  which are  $\Pi_1^1$ ?
- (5) Are there natural forcing notions other than Sacks and Miller to which the hypothesis of Theorem 3.10 applies?

## REFERENCES

1. Jörg Brendle, Lorenz Halbeisen, and Benedikt Löwe, *Silver measurability and its relation to other regularity properties*, Mathematical Proceedings of the Cambridge Philosophical Society **138** (2005), no. 01, 135–149.
2. Vera Fischer, Sy David Friedman, and Yurii Khomskii, *Co-analytic mad families and definable wellorders*, Archive for Mathematical Logic **52** (2013), no. 7-8, 809–822.
3. Vera Fischer, Sy David Friedman, and Asger Törnquist, *Projective maximal families of orthogonal measures with large continuum*, Journal of Logic and Analysis **4** (2012), Paper 9.
4. Vera Fischer, Sy David Friedman, and Lyubomyr Zdomskyy, *Projective wellorders and mad families with large continuum*, Annals of Pure and Applied Logic **162** (2011), no. 11, 853–862.
5. Vera Fischer and Asger Törnquist, *A co-analytic maximal set of orthogonal measures*, Journal of Symbolic Logic **75** (2010), no. 4, 1403–1414.
6. Sy David Friedman and Lyubomyr Zdomskyy, *Projective mad families*, Annals of Pure and Applied Logic **161** (2010), no. 12, 1581–1587.
7. Lorenz J. Halbeisen, *Combinatorial set theory*, Springer Monographs in Mathematics, Springer, 2012.
8. Leo Harrington,  $\Pi_2^1$  sets and  $\Pi_2^1$  singletons, Proceedings of the American Mathematical Society **52** (1975), 356–360.
9. G. Hjorth, *Classification and orbit equivalence relations*, Mathematical Surveys and Monographs, vol. 75, American Mathematical Society, 2000.
10. Jaime I. Ihoda and Saharon Shelah,  $\Delta_2^1$ -sets of reals, Annals of Pure and Applied Logic **42** (1989), no. 3, 207–223.
11. Thomas Jech, *Set theory*, third millenium ed., Springer monographs in mathematics, Springer, 2003.
12. Shizuo Kakutani, *On equivalence of infinite product measures*, Annals of Mathematics **49** (1948), no. 1, pp. 214–224.
13. Alexander S Kechris, *On a notion of smallness for subsets of the Baire space*, Transactions of the American Mathematical Society **229** (1977), 191–207.
14. Alexander S. Kechris, *Classical descriptive set theory*, Graduate texts in mathematics, no. 156, Springer-Verlag, 1995.
15. Alexander S. Kechris and Nikolaos E. Sofronidis, *A strong generic ergodicity property of unitary and self-adjoint operators*, Ergodic Theory and Dynamical Systems **21** (2001), 1459–1479.
16. Richard Mansfield and Galen Weitkamp, *Recursive aspects of descriptive set theory*, Oxford Logic Guides, vol. 11, Oxford University Press, 1985.

17. Adrian R. D. Mathias, *Happy families*, Annals of Mathematical logic **12** (1977), no. 1, 59–111.
18. Benjamin D. Miller, *Definable transversals of analytic equivalence relations*, preprint available at the authors website.
19. Yiannis N. Moschovakis, *Descriptive set theory*, Mathematical Surveys and Monographs, no. 155, American Mathematical Society, 2009.
20. David Preiss and Jan Rataj, *Maximal sets of orthogonal measures are not analytic*, Proceedings of the American Mathematical Society **93** (1985), no. 3, pp. 471–476.
21. Dimitri Shlyakhtenko, *On the classification of full factors of type III*, Transactions of the American Mathematical Society **356** (2004), no. 10, 4143–4159.
22. Stevo Todorćević, *Introduction to Ramsey spaces*, Princeton University Press, 2010.
23. Asger Törnquist, *Conjugacy, orbit equivalence and classification of measure-preserving group actions*, Ergodic Theory and Dynamical Systems **29** (2009), no. 3, 1033–1049.
24. Asger Törnquist,  $\Sigma_2^1$  and  $\Pi_1^1$  mad families, Journal of Symbolic Logic **78** (2013), no. 4, 1181–1182.
25. Asger Törnquist, *Definability and almost disjoint families*, arXiv:1503.07577 [math.LO], 2015.
26. Asger Törnquist and William Weiss, *Definable Davies’ theorem*, Fundamenta Mathematicae **205** (2009), no. 1, 77–89.
27. Jindřich Zapletal, *Forcing idealized*, Cambridge Tracts in Mathematics, vol. 174, Cambridge University Press, 2008.

DEPARTMENT OF MATHEMATICAL SCIENCES, UNIVERSITY OF COPENHAGEN, UNIVERSITETSPARKEN 5, 2100 COPENHAGEN, DENMARK

*E-mail address:* david.s@math.ku.dk

*E-mail address:* asgert@math.ku.dk